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ORTHOGONAL DECOMPOSITION OF SOME AFFINE LIE ALGEBRAS IN TERMS OF THEIR HEISENBERG SUBALGEBRAS

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Abstract

In the present note we suggest an affinization of a theorem by Kostrikin et.al. about the decomposition of some complex simple Lie algebras $\mathcal{G}$ into the algebraic sum of pairwise orthogonal Cartan subalgebras. We point out that the untwisted affine Kac-Moody algebras of types A_{p^m-1} (p prime, $m \geq 1$), B_r, C_{2^m}, D_r, G_2, E_7, E_8 can be decomposed into the algebraic sum of pairwise orthogonal Heisenberg subalgebras. The A_{p^m-1} and G_2 cases are discussed in great detail. Some possible applications of such decompositions are also discussed.

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1 Introduction

In the early eighties A. I. Kostrikin, I. A. Kostrikin and V. A. Ufnarovskii proved a theorem [1], see also [2] and [3], that the complex Lie algebras $\mathcal{G}$ of types A_{p^m-1} (p prime, $m \geq 1$), B_r, C_{2^m}, D_r, G_2, E_7, E_8 can be decomposed into the algebraic sum of Cartan subalgebras, pairwise orthogonal with respect to the Killing form; i.e.

$$\mathcal{G} = \mathcal{H}_0 \oplus \mathcal{H}_1 \oplus \cdots \oplus \mathcal{H}_h \tag{1}$$

with

$$\mathrm{Tr}\,(\mathcal{H}_i\,\mathcal{H}_j) = 0 \quad \text{for } i \neq j, \tag{2}$$

where h is the Coxeter number of $\mathcal{G}$. The fact that the number of Cartan subalgebras in the above decomposition (1) is $h+1$ accords with the well-known observation that for all semisimple algebras $h+1 = \dim\,\mathcal{G}/r,\; r = \mathrm{rank}\;\mathcal{G}$.

Here we show that this theorem can be extended to many of the corresponding affine Lie algebras $\hat{\mathcal{G}}$, in the sense that the latter may be decomposed into the algebraic sum of pairwise orthogonal Heisenberg subalgebras. Those subalgebras are in fact the affinization of the Cartan subalgebras in the corresponding decomposition of the finite–dimensional algebras $\mathcal{G}$. The problem of extending the results to the affine case consists in identifying the gradations of the Kac-Moody algebra $\hat{\mathcal{G}}$ which respect the decomposition (1).

The integral gradations of an affine Kac-Moody algebra $\hat{\mathcal{G}}$ are provided by the grading operators [4]

$$Q = \sum_{a=1}^{r} s_a \lambda_a^v \cdot H + Nd, \tag{3}$$

where $(s_0, s_1, \cdots, s_r)$ is a vector of non negative, relatively prime integers, $\lambda_a^v \equiv 2\lambda_a/\alpha_a^2$ with λ_a and α_a being the fundamental weights and simple roots of $\mathcal{G}$ respectively. Furthermore,

$$N = \sum_{i=0}^{r} s_i\, m_i^{\psi}, \qquad \psi = \sum_{a=1}^{r} m_a^{\psi}\, \alpha_a, \qquad m_0^{\psi} \equiv 1 \tag{4}$$

and ψ is the maximal root of $\mathcal{G}$. Two gradations are counted as equivalent if the corresponding vectors $(s_0, s_1, \cdots, s_r)$ and $(s_0', s_1', \cdots s_r')$ are related by a symmetry of the extended Dynkin diagram of $\mathcal{G}$.

Given a finite–dimensional complex simple Lie algebra $\mathcal{G}$ with decomposition (1), we look for a gradation such that the corresponding affine Kac–Moody algebra $\hat{\mathcal{G}}$ can be written as

$$\hat{\mathcal{G}} = \hat{\mathcal{H}}_0 \oplus \hat{\mathcal{H}}_1 \oplus \cdots \oplus \hat{\mathcal{H}}_h, \tag{5}$$

where each subspace $\hat{\mathcal{H}}_i$ is a Heisenberg subalgebra of $\hat{\mathcal{G}}$ with generators being eigenvectors of the corresponding grading operator Q. In addition, under the restriction of $\hat{\mathcal{G}}$ to $\mathcal{G}$, the elements of $\hat{\mathcal{H}}_i$ become those of $\mathcal{H}_i$.

For the case of the homogeneous gradation it is quite trivial to make the extension of the decomposition (1) with the orthogonality property (2) to the affine case. In this case $s_0 = 1$ and $s_a = 0$ for $a = 1, 2 \cdots, r$, and so $Q = d$. This gradation is insensitive to the structure of the root system of $\mathcal{G}$ and therefore it does not matter what linear combinations of the Weyl–Cartan basis elements the generators of $\mathcal{H}_i$ are. We construct the generators of $\hat{\mathcal{H}}_i$ by replacing in $\mathcal{H}_i$ the elements H_a

by H_a^n and E_α by E_α^n, with n being the integral eigenvalues of d; in other words $[d, H_a^n] = nH_a^n$, and $[d, E_\alpha^n] = nE_\alpha^n$. The orthogonality condition is preserved since if, say S and T are two orthogonal elements of $\mathcal{G}$, so are the elements S^m and T^n of $\hat{\mathcal{G}}$; i.e., $\mathrm{Tr}(S^m T^n) = \mathrm{Tr}(ST)\,\delta_{m+n,0}$.

For other gradations the extension is more delicate, and we discuss here the case of the Lie algebras A_{p^m-1} and G_2 only. The cases of gradations which are not of type (3) deserve further study. At the end of section 2 we comment on a special gradation of A_2.

2 The case A_{p-1} with p prime

In refs. [1], [2] (see also [5]) the algebra A_{p-1} was decomposed as in (1) using some special properties of the defining representation. Consider the $p \times p$ matrices

$$g = \begin{pmatrix} 0 & x_1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & x_2 & 0 & \cdots & 0 \\ 0 & 0 & 0 & x_3 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & x_{p-1} \\ x_0 & 0 & 0 & 0 & \cdots & 0 \end{pmatrix}, \qquad D = \begin{pmatrix} 1 & 0 & 0 & 0 & \cdots & 0 \\ 0 & \omega & 0 & 0 & \cdots & 0 \\ 0 & 0 & \omega^2 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \omega^{p-2} & 0 \\ 0 & 0 & 0 & 0 & \cdots & \omega^{p-1} \end{pmatrix}, \tag{6}$$

where x_i, $i = 0, 1, 2 \ldots p-1$ are arbitrary non vanishing constants, ω is the p-th root of unit ($\omega = \exp(2\pi i/p)$, $\omega^p = 1$), and so

$$g\,D = \omega\,D\,g, \qquad g^p = x_0 x_1 \ldots x_{p-1} \mathbb{1}, \qquad D^p = \mathbb{1}. \tag{7}$$

The subspaces in (1) are defined as

$$\begin{aligned} \mathcal{H}_0 &= \{D^k\,,\ k = 1, 2, 3, \ldots, p-1\}; \\ \mathcal{H}_{l+1} &= \{\left(D^l g\right)^k\,,\ k = 1, 2, 3, \ldots, p-1\}, \qquad l = 0, 1, 2, \ldots, p-1; \end{aligned} \tag{8}$$

and the Coxeter number h of A_{p-1} equals p.

The subspaces (8) are obviously abelian. However, the linear independency and orthogonality of them requires p to be a prime number. Using (7) one observes that

$$\left(D^l g\right)^k \sim D^{lk} g^k, \tag{9}$$

where here, and below, the symbol $\sim$ means the two quantities are proportional (the proportionality constant being ω dependent). Therefore, if for given l and l' one has $lk = l'k + mp$, with m integer, then $\left(D^l g\right)^k \sim \left(D^{l'} g\right)^k$. Now, if p is prime, either k or $(l - l')$ will have to have p in its expansion as a product of primes. But this can not happen since they can be at most $(p-1)$. Therefore the subspaces (8) are linearly independent for p prime. As for the orthogonality, consider $\mathrm{Tr}\left(\left(D^l g\right)^k \left(D^{l'} g\right)^{k'}\right) \sim \mathrm{Tr}\left(D^{lk+l'k'} g^{k+k'}\right)$. That does not vanish for $l \neq l'$ only if $k+k' = mp$ and $lk+l'k' = np$, with m and n integers, and so for $(l-l')k = (n-ml')p$, which is impossible for p prime.

Now to extend the consideration to the affine case, we have to select a gradation which respects the decomposition written above. As we have already said, the

homogenous gradation fits our needs. However, in this particular case the principal gradation also does, and we find it more interesting and important for applications (see comments at the end of this section on an additional gradation for A_2).

In fact the homogeneous and principal gradations are the only integer gradations which respect the decomposition above. In order to see that, one observes that the elements $D^l g$ are linear combinations of the step operators for simple roots E_{α_a}, $a = 1, 2, \ldots, p-1$, and $E_{-\psi}$ where ψ is the highest root. Now when we affinize the algebra we will need to find integers n_a and n_b such that $E_{\alpha_a}^{n_a}$ and $E_{\alpha_b}^{n_b}$ have the same grade under the grading operator (3). Therefore we need $n_b = n_a + \frac{s_a - s_b}{N}$. But since s_a and s_b are non negative it follows that $\mid s_a - s_b \mid$ is smaller than s_a and/or s_b. On the other hand, from (4) we see that N is greater than s_a and s_b. So it is impossible to satisfy the above equality with n_a and n_b integers unless $s_a = s_b$ for all a and b. Now we also need $E_{\alpha_a}^{n_a}$ and $E_{-\psi}^{n_0}$ to have the same grade, and so $s_a + N n_a = -\sum_{b=1}^{p-1} s_b + N n_0$ (since $\psi = \sum_{b=1}^{p-1} \alpha_b$). Therefore we need $n_a = n_0 - \frac{ps}{(p-1)s + s_0}$, where we used the fact that all s_a's are equal and have set $s_a \equiv s$ for any a. So we have to have $s = 0$ (homogeneous) or $s = s_0$ (principal).

We then consider the principal gradation and define the following generators of the affine Kac–Moody algebra $\hat{sl}(p, \mathbb{C})$:

$$\hat{\mathcal{H}}^0_{k,pn} \equiv \omega_k H_1^n + \omega_k^2 H_2^n + \ldots + \omega_k^{p-1} H_{p-1}^n + H_0^n, \tag{10}$$

with $k = 1, 2, \ldots, p-1$; and also

$$\begin{aligned}
\hat{\mathcal{H}}^{l+1}_{1,pn+1} &\equiv \omega_l E^n_{\alpha_1} + \omega_l^2 E^n_{\alpha_2} + \omega_l^3 E^n_{\alpha_3} + \ldots + \omega_l^{p-1} E^n_{\alpha_{p-1}} + E^{n+1}_{\alpha_0}; && (11)\\
\hat{\mathcal{H}}^{l+1}_{2,pn+2} &\equiv \omega_{2l} E^n_{\alpha_1+\alpha_2} + \omega_{2l}^2 E^n_{\alpha_2+\alpha_3} + \omega_{2l}^3 E^n_{\alpha_3+\alpha_4} + \ldots \\
&+ \omega_{2l}^{p-2} E^n_{\alpha_{p-2}+\alpha_{p-1}} + \omega_{2l}^{p-1} E^{n+1}_{\alpha_{p-1}+\alpha_0} + E^{n+1}_{\alpha_0+\alpha_1}; && (12)\\
\hat{\mathcal{H}}^{l+1}_{3,pn+3} &\equiv \omega_{3l} E^n_{\alpha_1+\alpha_2+\alpha_3} + \omega_{3l}^2 E^n_{\alpha_2+\alpha_3+\alpha_4} + \omega_{3l}^3 E^n_{\alpha_3+\alpha_4+\alpha_5} + \ldots \\
&+ \omega_{3l}^{p-3} E^n_{\alpha_{p-3}+\alpha_{p-2}+\alpha_{p-1}} + \omega_{3l}^{p-2} E^{n+1}_{\alpha_{p-2}+\alpha_{p-1}+\alpha_0} \\
&+ \omega_{3l}^{p-1} E^{n+1}_{\alpha_{p-1}+\alpha_0+\alpha_1} + E^{n+1}_{\alpha_0+\alpha_1+\alpha_2}; && (13)\\
&\vdots \quad \vdots \quad \vdots \\
\hat{\mathcal{H}}^{l+1}_{k,pn+k} &\equiv \omega_{kl} E^n_{\alpha_1+\ldots+\alpha_k} + \omega_{kl}^2 E^n_{\alpha_2+\ldots+\alpha_{k+1}} + \omega_{kl}^3 E^n_{\alpha_3+\ldots+\alpha_{k+2}} + \ldots \\
&+ \omega_{kl}^{p-k} E^n_{\alpha_{p-k}+\ldots+\alpha_{p-1}} + \omega_{kl}^{p-k+1} E^{n+1}_{\alpha_{p-k+1}+\ldots+\alpha_{p-1}+\alpha_0} + \ldots \\
&+ \omega_{kl}^{p-1} E^{n+1}_{\alpha_{p-1}+\alpha_0+\alpha_1+\ldots+\alpha_{k-2}} + E^{n+1}_{\alpha_0+\alpha_1+\ldots+\alpha_{k-1}}; && (14)\\
&\vdots \quad \vdots \quad \vdots \\
\hat{\mathcal{H}}^{l+1}_{p-1,pn+p-1} &\equiv \omega_{(p-1)l} E^n_{\alpha_1+\alpha_2+\ldots+\alpha_{p-1}} + \omega_{(p-1)l}^2 E^{n+1}_{\alpha_2+\alpha_3+\ldots+\alpha_{p-1}+\alpha_0} \\
&+ \omega_{(p-1)l}^3 E^{n+1}_{\alpha_3+\ldots+\alpha_{p-1}+\alpha_0+\alpha_1} + \ldots \\
&+ \omega_{(p-1)l}^{p-1} E^{n+1}_{\alpha_{p-1}+\alpha_0+\alpha_1+\ldots+\alpha_{p-3}} + E^{n+1}_{\alpha_0+\alpha_1+\ldots+\alpha_{p-2}}, && (15)
\end{aligned}$$

with $l = 0, 1, 2, \ldots, p-1$. Here we have denoted $\omega_k \equiv \exp(2\pi i k/p)$, and α_i, $i = 0, 1, 2, \ldots, p-1$ are the simple roots of $\hat{sl}(p, \mathbb{C})$ with $\alpha_0 = -\psi$; $\psi \equiv$ is the maximal root of $\hat{sl}(p, \mathbb{C})$. The elements $H_i^n \equiv 2\alpha_i \cdot H^n / \alpha_i^2 + \delta_{i,0} C$ with C being the centre of $\hat{sl}(p, \mathbb{C})$, $i = 0, 1, 2, \ldots, p-1$ and E_α^n are the generators of $\hat{sl}(p, \mathbb{C})$ in the Chevalley basis. They satisfy the commutation relations

$$[H_a^m \,,\, H_b^n] \;=\; C \frac{2}{\alpha_a^2} K_{ab} m \delta_{m+n,0}$$

$$
\begin{aligned}
[H_a^m\,,\,E_{\pm\alpha}^n] &= \pm K_{\alpha a}E_{\pm\alpha}^{m+n}\\
[E_\alpha^m\,,\,E_{-\alpha}^n] &= l_a^\alpha H_a^{m+n} + C\frac{2}{\alpha^2}m\delta_{m+n,0}\\
[E_\alpha^m\,,\,E_\beta^n] &= \epsilon(\alpha,\beta)E_{\alpha+\beta}^{m+n}; \qquad \text{if } \alpha+\beta \text{ is a root of } \mathcal{G}\\
[d\,,\,H_b^m] &= mH_b^m\\
[d\,,\,E_\alpha^m] &= mE_\alpha^m
\end{aligned}
\tag{16}
$$

with $a, b = 1, 2, \ldots, p-1$; $K_{\alpha a} = 2\alpha\cdot\alpha_a/\alpha_a^2 = n_b^\alpha K_{ba}$ with K_{ab} being the Cartan matrix of $\mathcal{G}$; n_a^α and l_a^α are the integers in the expansions $\alpha = \sum_{a=1}^r n_a^\alpha \alpha_a$ and $\alpha/\alpha^2 = \sum_{a=1}^r l_a^\alpha \alpha_a/\alpha_a^2$, and $\epsilon(\alpha,\beta)$ are structure constants.

Notice all the generators defined above are eigenvectors of the principal gradation operator, see (3),

$$
Q_{\rm pr.} = \sum_{a=1}^{p-1} \lambda_a^v \cdot H + pd, \tag{17}
$$

i.e.

$$
[Q_{\rm pr.}\,,\,\hat{\mathcal{H}}_{k,pn}^0] = pn\,\hat{\mathcal{H}}_{k,pn}^0, \tag{18}
$$

$$
[Q_{\rm pr.}\,,\,\hat{\mathcal{H}}_{k,pn+k}^{l+1}] = (pn+k)\,\hat{\mathcal{H}}_{k,pn+k}^{l+1}. \tag{19}
$$

We then define the subspaces in (5) as

$$
\begin{aligned}
\hat{\mathcal{H}}_0 &\equiv \{\hat{\mathcal{H}}_{k,pn}^0\,,\,k = 1, 2, \ldots, p-1\,,\,n\in\mathbb{Z}\};\\
\hat{\mathcal{H}}_{l+1} &\equiv \{\hat{\mathcal{H}}_{k,pn+k}^{l+1}\,,\,k = 1, 2, \ldots, p-1\,,\,n\in\mathbb{Z}\} \qquad l = 0, 1, \ldots, p-1.
\end{aligned}
\tag{20}
$$

Each one of these subspaces constitutes a Heisenberg subalgebra of $\hat{sl}(p,\mathbb{C})$. The first one, $\hat{\mathcal{H}}_0$, is a homogeneous Heisenberg subalgebra and the others, $\hat{\mathcal{H}}_{l+1}$, are principal ones. The proof that this is indeed so, can be obtained by realizing the Kac-Moody algebra (16) in the following way.

Introduce the bracket

$$
[A(z)\,,\,B(z)] \equiv [A(z)\,,\,B(z)]_{\rm o.c.} + C\mathbb{1}\oint \frac{dz}{2\pi i}\,{\rm Tr}(B(z)\frac{d}{dz}A(z)) \tag{21}
$$

where $[\,,\,]_{\rm o.c.}$ is the ordinary commutator between matrices, i.e. $[X\,,\,Y]_{\rm o.c.} \equiv XY - YX$. The commutation relations (16) can be obtained from this bracket by writing the Kac-Moody generators as the matrices:

$$
\begin{aligned}
(H_a^n)_{ij} &\equiv z^n(\delta_{i,a}\delta_{j,a} - \delta_{i,a+1}\delta_{j,a+1}),\\
(H_0^n)_{ij} &\equiv z^n(-\delta_{i,1}\delta_{j,1} + \delta_{i,p}\delta_{j,p}) + C\delta_{n,0}\,\delta_{ij},\\
\left(E_{\alpha_a}^n\right)_{ij} &\equiv z^n\delta_{i,a}\delta_{j,a+1},\\
\left(E_{\alpha_0}^n\right)_{ij} &\equiv z^n\delta_{i,p}\delta_{j,1},\\
\left(E_{\alpha_a+\alpha_{a+1}+\ldots+\alpha_{a+k}}^n\right)_{ij} &\equiv z^n\delta_{i,a}\delta_{j,a+k+1}
\end{aligned}
\tag{22}
$$

with $i, j = 1, 2, \ldots p$; $a = 1, 2, \ldots p-1$ and $k = 1, 2, \ldots p-a-1$. The negative root step operators are obtained from the positives ones by transposing the matrices and inverting the power of z, i.e. $E_{-\alpha}^n = (E_\alpha^{-n})^\dagger$ (notice that for $k > l$, $\alpha_k + \alpha_{k+1} + \ldots +$

$\alpha_{p-1} + \alpha_0 + \alpha_1 + \alpha_2 + \ldots + \alpha_l = -\alpha_{l+1} - \alpha_{l+2} - \ldots - \alpha_{k-1}$). In addition we have $d \equiv \mathbb{1}\, z\frac{d}{dz}$.

By introducing the matrix

$$g(z) \equiv g\mid_{x_0=z, x_a=1} \equiv E^0_{\alpha_1} + E^0_{\alpha_2} + \ldots + E^0_{\alpha_{p-1}} + E^1_{\alpha_0} \tag{23}$$

with $a = 1, 2, \ldots p-1$ and g given in (6) ($g(z)^p = z\mathbb{1}$), we can then write the generators (10)-(15) as

$$\begin{aligned} \hat{\mathcal{H}}^0_{k,pn} &\equiv z^n\,(\omega_k - 1)\,D^k + \delta_{n,0} C\mathbb{1} \\ \hat{\mathcal{H}}^{l+1}_{k,pn+k} &\equiv z^n \omega_{kl} D^{kl} (g(z))^k \sim z^n \left(D^l g(z)\right)^k \end{aligned} \tag{24}$$

with $l = 0, 1, \ldots p-1$ and $k = 1, 2, \ldots p-1$.

The orthogonality of the subspaces (20) is shown by using the same arguments as the ones given above in the finite dimensional case, but now using the bilinear form

$$\langle A(z) \mid B(z) \rangle \equiv \oint \frac{dz}{2\pi i z}\, \mathrm{Tr}(A(z)B(z)) \tag{25}$$

Using (21) one obtains

$$[\hat{\mathcal{H}}^0_{j,pm}\,,\,\hat{\mathcal{H}}^0_{k,pn}] = C\,(\omega_j - 1)(\omega_{p-j} - 1)\,pm\,\delta_{n+m,0}\delta_{k,p-j} \tag{26}$$

$$[\hat{\mathcal{H}}^{l+1}_{j,pm+j}\,,\,\hat{\mathcal{H}}^{l+1}_{k,pn+k}] = C\,\omega_l^{k+j+kj}\,(pm+j)\,\delta_{m+n+1,0}\delta_{k,p-j} \tag{27}$$

which constitute a set of $p+1$ orthogonal Heisenberg subalgebras.

We now comment on a connection between the decomposition above and a special basis for the affine Kac-Moody algebras. Such a basis [6, 7] is a generalization of the basis constructed by Kostant [8] for finite dimesional simple Lie algebras. Consider the principal gradation of an affine Kac-Moody algebra $\hat{\mathcal{G}}$

$$\hat{\mathcal{G}} = \oplus\, \hat{\mathcal{G}}_n \tag{28}$$

with $n \in \mathbb{Z}$ and

$$[Q_{\mathrm{pr.}}\,,\,\hat{\mathcal{G}}_n] = n\,\hat{\mathcal{G}}_n \tag{29}$$

and $Q_{\mathrm{pr.}}$ being the grading operator (3) corresponding to all s_i's equal to 1. Let $\hat{\mathcal{H}} \subset \hat{\mathcal{G}}$ be a principal Heisenberg subalgebra with generators $E_M \in \hat{\mathcal{G}}_M$, where M are the exponents of $\hat{\mathcal{G}}$, and satisfying

$$[E_M\,,\,E_N] = \frac{C}{h}\,\mathrm{Tr}(E_M E_N) M \delta_{M+N,0} \tag{30}$$

These exponents have a period equal to the Coxeter number h of the finite simple Lie algebra $\mathcal{G}$ (of which $\hat{\mathcal{G}}$ is the affinization of), i.e. they have the form $M = m_a + nh$ where n is an integer and m_a, $a = 1, 2, \ldots r$ are the exponents of $\mathcal{G}$ [7]. In particular, unity is always an exponent and it follows that E_1 is a linear combination of the simple root step operators $E^0_{\alpha_a}$ and $E^1_{\alpha_0}$, $a = 1, 2, \ldots r$. The complement $\mathcal{F}$ of $\hat{\mathcal{H}}$ in $\hat{\mathcal{G}}$ is such that the dimension of $\mathcal{F}_m \subset \hat{\mathcal{G}}_m$ for $m \neq 0$ is equal to the rank of $\mathcal{G}$ ($\equiv r$). The subspace $\mathcal{F}_0$ has dimension $r+2$ and is generated by C, D and H^0_a, $a = 1, 2, \ldots, r$. The complement $\mathcal{F}$ ad-diagonalize $\hat{\mathcal{H}}$ and except for the extra two generators of $\mathcal{F}_0$, the basis, F^a_m, of $\mathcal{F}_m$ can be chosen such that

$$[E_M\,,\,F^a_n] \sim F^a_{M+n} \tag{31}$$

We point out that for the affine algebras $\hat{sl}(p,\mathbb{C})$ with p prime, the complement $\mathcal{F}$ is in fact an algebraic sum of p Heisenberg subalgebras, with $p-1$ of them being principal and 1 being homogeneous. In addition, the role played by $\hat{\mathcal{H}}$ can be replaced by any one of these $p-1$ principal Heisenberg subalgebras, with $\hat{\mathcal{H}}$ being now part of the new complement $\mathcal{F}$.

To establish this fact we show that by choosing one the principal Heisenberg subalgebras $\hat{\mathcal{H}}_{l+1}$ in (20) to play the role of $\hat{\mathcal{H}}$ as above, then the basis elements of the remaining Heisenberg subalgebras ad-diagonalize $\hat{\mathcal{H}}$.

Using (21) and (24) one gets

$$[\hat{\mathcal{H}}^{l+1}_{j,pm+j}\,,\,\hat{\mathcal{H}}^{0}_{k,pn}] = (\omega_{kj}-1)(1-\omega_{p-k})\hat{\mathcal{H}}^{l'+1}_{j,p(m+n)+j} \tag{32}$$

where l' is the integer such that $(j(l'-l)-k)$ is a multiple of p. Notice that l' and l are never equal since k is not allowed to be a multiple of p (see (24)). In addition one gets ($l\neq l'$)

$$[\hat{\mathcal{H}}^{l+1}_{j,pm+j}\,,\,\hat{\mathcal{H}}^{l'+1}_{k,pn+k}] = (\omega^{jk}_{l'}-\omega^{jk}_{l})\hat{\mathcal{H}}^{l''+1}_{k+j,p(m+n)+k+j} \qquad \text{if } k+j\neq p \tag{33}$$

where l'' is the integer such that $k(l''-l')+j(l''-l)$ is a multiple of p, and (again $l\neq l'$)

$$[\hat{\mathcal{H}}^{l+1}_{j,pm+j}\,,\,\hat{\mathcal{H}}^{l'+1}_{k,pn+k}] = \frac{\omega_{j(l-l')}(\omega^{-j^2}_{l'}-\omega^{-j^2}_{l})}{(\omega_{j(l-l')}-1)}\hat{\mathcal{H}}^{0}_{j(l-l'),p(m+n+1)} \qquad \text{if } k+j=p \tag{34}$$

Notice that in (33) l'' can not be equal to l or l', since if it is equal, let us say, to l then either k or $(l''-l')$ would have to have p in its expansion as a product of primes, and that is not allowed. We point out that the term $\delta_{n,0}\,C\,\mathbb{1}$ in the expression of $\hat{\mathcal{H}}^{0}_{k,pn}$ given by (24), was not important in obtaining (26) or the orthogonality property. However it was crucial in establishing (34).

Therefore the elements of the $p+1$ Heisenberg subalgebras (20) ad-diagonalize any one of the p principal Heisenberg subalgebras $\hat{\mathcal{H}}_{l+1}$, $l=0,1,\ldots p-1$. They therefore constitute a basis for the complement $\mathcal{F}$ discussed above. Notice however that the basis elements F^a_n for a fixed a are not all in the same Heisenberg subalgebra. In addition, the results of brackets between elements of two given Heisenberg subalgebras do not belong necessarily to a fixed third Heisenberg subalgebra, in other words the decomposition is not multiplicative orthogonal in the sense of ref. [3].

We end this section by making a comment on a fine grading of A_2 [5]. It is a $\mathbb{Z}_3\times\mathbb{Z}_3$ gradation where the generators have the following gradings: $D=(1,0)$, $D^2=(2,0)$, $g=(0,2)$, $g^2=(0,1)$, $Dg=(1,2)$, $(Dg)^2=(2,1)$, $D^2g=(2,2)$,and $(D^2g)^2=(1,1)$, with D and g being the 3×3 matrices defined in (6). The addition of the gradings is given by the sum mod 3. The operator with grade $(0,0)$ is the identity matrix, which is not part of the algebra A_2 (one has to extend to $gl(3)$ to incorporate it). Since this gradation respects the decomposotion (1) for A_2 one can elevate it to the corresponding affine algebra. One can just attach an integer index n to the generators (eigenvalue of the derivation d, see (16)) and get a gradation $\mathbb{Z}_3\times\mathbb{Z}_3\times\mathbb{Z}$ of the Kac-Moody algebra $A_2^{(1)}$. Such a gradation is not of the type (3). However, the decomposition in terms of Heisenberg subalgebras (5) we get in this case is the same as the one we would get with the homogeneous gradation.

3 The case A_{p^m-1} with p prime and $m>1$

In refs. [1], [2] a construction of the orthogonal decompositions for the algebras A_{p^m-1} with p prime and $m>1$ was also performed. In the case $m=1$ (see previous section) the basic idea was that matrices made of powers of a given matrix M commute among themselves. For the case $m>1$ one uses the fact that, given the matrices M_i, $i=1,2,\ldots,m$, then the quantities $M_1^{n_1}\otimes M_2^{n_2}\otimes\ldots\otimes M_m^{n_m}$, commute among themselves under the usual commutator in the tensor product

$$[A_1\otimes\ldots\otimes A_m\,,\,B_1\otimes\ldots\otimes B_m]\equiv A_1B_1\otimes\ldots\otimes A_mB_m-B_1A_1\otimes\ldots\otimes B_mA_m \tag{35}$$

Therefore for the complex Lie algebra A_{p^m-1} with p prime and $m>1$ we introduce the following subspaces

$$\begin{aligned}\mathcal{H}_0 &\equiv \{D^{k_1}\otimes D^{k_2}\otimes\ldots\otimes D^{k_m}\mid k_i=0,1,2,\ldots,p-1\}\\ \mathcal{H}_{\mathbf{L}} &\equiv \{(D^{l_1}g)^{k_1}\otimes(D^{l_2}g)^{k_2}\otimes\ldots\otimes(D^{l_m}g)^{k_m}\mid k_i=0,1,2,\ldots,p-1\}\end{aligned} \tag{36}$$

where D and g are the $p\times p$ matrices defined in (6), and $\mathbf{L}\equiv(l_1,l_2,\ldots l_m)$, with $l_i=0,1,2,\ldots,p-1$. In addition the k_i's are not all allowed to have the value zero.

Notice that there are p^m possible vectors $\mathbf{L}$ and that the dimension of $\mathcal{H}_0$ and $\mathcal{H}_{\mathbf{a}}$ is p^m-1. Therefore we have $(p^m-1)(p^m+1)$ generators in those subspaces, which is the dimension of A_{p^m-1}. Each one the above subspaces is obviously abelian. For p being a prime number those generators are linearly independent and constitute a basis for A_{p^m-1}. The proof for that and the orthogonality property is very similar for the case $m=1$, given in the previous section, and for more details we refer to [2].

In order to extend the above decomposition to the corresponding affine Kac-Moody algebra we have to find a gradation which respects the decomposition in terms of (36). We now show that the only two allowed gradations are given by

$$\begin{aligned}\mathbf{s}_{\rm hom} &= (1,0,0,\ldots,0)\,, \qquad \text{homogeneous gradation;}\\ \mathbf{s}_{\rm p} &= \left(1,\underbrace{0,\ldots,0,1}_{p\text{ entries}},\underbrace{0,\ldots,0,1}_{p\text{ entries}},\ldots\ldots\underbrace{0,\ldots,0,1}_{p\text{ entries}},\underbrace{0,\ldots,0}_{p-1\text{ entries}}\right),\end{aligned} \tag{37}$$

where we have denoted the integers s_a in (3) as the components of the vector $\mathbf{s}\equiv(s_0,s_1,\ldots,s_{p^m-1})$.

To show that, it is better to express the elements of the subspaces (36) in terms of step operators. The elements of $\mathcal{H}_0$ are diagonal and therefore are linear combinations of the Cartan subalgebra generators only. Therefore, under (3), they will be sensitive to the d term only. Consequently, when affinizing the algebra what one has to do is to make sure all Cartan subalgebras generators appearing in the expansion of the affinized version of a given element of $\mathcal{H}_0$ have the same d-eigenvalue. Therefore $\mathcal{H}_0$ does not impose any restrictions on the possible gradations.

As for the elements of $\mathcal{H}_{\mathbf{L}}$ we can write, using (7), $(D^{l_1}g)^{k_1}\otimes(D^{l_2}g)^{k_2}\otimes\ldots\otimes(D^{l_m}g)^{k_m}\sim(D^{l_1k_1}\otimes D^{l_2k_2}\otimes\ldots\otimes D^{l_mk_m})(g^{k_1}\otimes g^{k_2}\otimes\ldots\otimes g^{k_m})$. When expanding such an element in the Chevalley basis, the step operator content is determined by the term $(g^{k_1}\otimes g^{k_2}\otimes\ldots\otimes g^{k_m})$, since the other one just dictates which linear combination of these step operators it corresponds to. Therefore in order to determine the gradings respecting the decomposition (36), one has to consider only the g part of those elements only.

Consider the quantity $\mathbb{1} \otimes \mathbb{1} \otimes \ldots \otimes \mathbb{1} \otimes g$, which, written in terms of $p^m \times p^m$ matrices, has the form

$$\mathbb{1} \otimes \mathbb{1} \otimes \ldots \otimes \mathbb{1} \otimes g = \begin{pmatrix} g & 0 & 0 & \cdots & 0 \\ 0 & g & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & g \end{pmatrix} \tag{38}$$

where g is the matrix given in (6). Using the matrices for step operators in the defining representation of A_{p^m-1}, given in (22) (excluding the z terms), one concludes that the above element contains in its expansion all the step operators for simple roots E_{α_a}, except those corresponding to $a = kp$, with $k = 1, 2, \ldots p^{m-1} - 1$. When affinizing such an element we will need to find integers n_a and n_b such that $E_{\alpha_a}^{n_a}$ and $E_{\alpha_b}^{n_b}$ have the same grade under the grading operator (3). Using arguments similar to those of the previous section (see paragraph before eq. (10)) one then concludes that the gradings respecting the decomposition (36) are such that $s_a = s_b$ for all a and b which are not multiples of p.

From (38), (6) and (22) one observes that $\mathbb{1} \otimes \mathbb{1} \otimes \ldots \otimes \mathbb{1} \otimes g$ also contains in its expansion step operators corresponding to the lowest root of each block, namely $E_{-\alpha_{jp+1}-\alpha_{jp+2}-\ldots-\alpha_{jp+p-1}}$, $j = 0, 1, \ldots p^{m-1} - 1$. Therefore we have to find integers n and $\bar{n}$ such that $E_{\alpha_{jp+s}}^{n}$ $(s = 1, 2, \ldots p-1)$ and $E_{-\alpha_{jp+1}-\alpha_{jp+2}-\ldots-\alpha_{jp+p-1}}^{\bar{n}}$ have the same grade under the grading operator (3). Therefore we need $s_{jp+s} + Nn = -\sum_{q=1}^{p-1} s_{jp+q} + N\bar{n}$, or $\bar{n} = n + \frac{ps_{jp+s}}{N}$, where we have used the fact that, according to the reasoning above, all s_{jp+q}'s $(q = 1, 2, \ldots p-1)$ are equal. But, by the same reasons, we have $N = s_0 + p^{m-1}(p-1)s_{jp+s} + \sum_{k=1}^{p^{m-1}-1} s_{kp}$. Therefore, since all s_i's are non negative, it follows that $\frac{ps_{jp+s}}{N}$ can not be an integer unless

$$s_{jp+s} = 0, \qquad \text{for } j = 0, 1, \ldots p^{m-1} - 1 \text{ and } s = 1, 2, \ldots p-1 \tag{39}$$

The case $p = 2$ with $m = 2$ may lead to other possibilities, but one can check that due to the symmetries of the A_3 Dynkin diagram they do not lead to gradings inequivalent to those given in (37).

Consider now the quantity

$$\mathbb{1} \otimes \mathbb{1} \otimes \ldots \otimes \mathbb{1} \otimes g \otimes g = \begin{pmatrix} 0 & g & 0 & \cdots & 0 \\ 0 & 0 & g & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & g \\ g & 0 & 0 & \cdots & 0 \end{pmatrix} \tag{40}$$

where again g is the matrix given in (6), but now to simplify we take all x_i's equal to unity. Using (22) one observes that such an element contains in its expansion all step operators for simple roots not appering in (38), namely $E_{\alpha_{kp}}$ with $k = 1, 2, \ldots p^{m-1} - 1$. Therefore we need to find integers n_k and $n_{\bar{k}}$ such that $E_{\alpha_{kp}}^{n^k}$ and $E_{\alpha_{\bar{k}p}}^{n_{\bar{k}}}$ have the same grade under the grading operator (3). Using the same arguments as above one concludes that all s_{kp}'s $(k = 1, 2, \ldots p^{m-1} - 1)$ must be equal.

One observes that the step operator for the lowest root of A_{p^m-1} also appears in (40). Therefore we need integers n_k and n_ψ such that $E_{\alpha_{kp}}^{n_k}$ and $E_{-\psi}^{n_\psi}$ $(\psi = \sum_{a=1}^{p^m-1} \alpha_a)$ have the same grade under the grading operator (3). So, we need $n_\psi = n_k + \frac{p^{m-1}s_{kp}}{N}$,

where $N = s_0 + (p^{m-1} - 1)s_{kp}$, and where we have used (39) and the fact that all s_{kp}'s are equal.One then concludes there are only two ways for the ratio $\frac{p^{m-1}s_{kp}}{N}$ to be an integer, namely

$$s_{kp} = 0 \qquad \text{for } k = 1, 2, \ldots p^{m-1} - 1 \tag{41}$$

or

$$s_{kp} = s_0 \qquad \text{for } k = 1, 2, \ldots p^{m-1} - 1 \tag{42}$$

Again the case $p = 2$ with $m = 2$ leads to other possibilities but they do not give inequivalent gradations. But those possibilities correspond exactly to the two gradations given in (37). One can check that by considering the other elements of $\mathcal{H}_{\mathbf{L}}$ one does not get any new restrictions on the allowed gradations, showing that (37) are the only two possibilities.

Therefore, if we use the homogeneous gradation $\mathbf{s}_{\text{hom}}$ of (37), the subspaces (36) will give rise to $p^m + 1$ homogeneous Heisenberg subalgebras. On the other hand if we use the gradation $\mathbf{s}_{\text{p}}$, the subspace $\mathcal{H}_0$ will lead to a homogeneous Heisenberg subalgebra, and the subspaces $\mathcal{H}_{\mathbf{L}}$ to p^m Heisenberg subalgebras associated to the grading $\mathbf{s}_{\text{p}}$.

The process of writing the elements of the subspaces (36) in the Chevalley basis is quite involved. However, there are some general patterns which are useful. As we commented in the paragraph before (38), the step operator content of those elements is determined by the terms $g^{k_1} \otimes g^{k_2} \otimes \ldots \otimes g^{k_m}$. These, when written in terms of $p^m \times p^m$ matrices, will be made of blocks which are powers of the $p \times p$ matrix g introduced in (6). Using (22) one can check that such a structure implies the step operators appearing in the expasion of a given element of the subspaces $\mathcal{H}_{\mathbf{L}}$ in (36) can be arranged in sets of p step operators such that the corresponding roots add up to zero. Below we discuss the cases A_{p^m-1} with $m = 2$ and $p = 2, 3$.

3.1 The case A_3

We denote the three simple roots of A_3 as α_a, $a = 1, 2, 3$, such that $2\alpha_1 \cdot \alpha_2/\alpha_2^2 = 2\alpha_3 \cdot \alpha_2/\alpha_2^2 = -1$ and $\alpha_1 \cdot \alpha_3 = 0$. We consider here the gradation $\mathbf{s}_{\text{p}}$ in (37), i.e.

$$\mathbf{s}_{\text{p}} = (1, 0, 1, 0) \tag{43}$$

The corresponding results for the homogeneous gradation can be easily obtained from those for $\mathbf{s}_{\text{p}}$. The generators for the affine subspace $\mathcal{H}_0$ in (36) are obviously H_a^n, $a = 1, 2, 3$. The commutativity of the subspaces $\mathcal{H}_{\mathbf{L}}$ and their pairwise orthogonality condition impose very rigid constraints on the allowed combinations of step operators. Now it is a quite simple algebraic task to convince oneself that the remaining 4 affine subspaces $\mathcal{H}_{\mathbf{L}}$ are

$$\begin{aligned}
\hat{\mathcal{H}}_1 \;&=\; \{(E^n_{\alpha_1} + E^n_{-\alpha_1}) + (E^n_{\alpha_3} + E^n_{-\alpha_3}),\ (E^n_{\alpha_1+\alpha_2} + E^{n+1}_{-\alpha_1-\alpha_2}) + (E^n_{\alpha_2+\alpha_3} + E^{n+1}_{-\alpha_2-\alpha_3}),\\
&\quad (E^n_{\alpha_1+\alpha_2+\alpha_3} + E^{n+1}_{-\alpha_1-\alpha_2-\alpha_3}) + (E^n_{\alpha_2} + E^{n+1}_{-\alpha_2})\};\\
\hat{\mathcal{H}}_2 \;&=\; \{(E^n_{\alpha_1} + E^n_{-\alpha_1}) - (E^n_{\alpha_3} + E^n_{-\alpha_3}),\ (E^n_{\alpha_1+\alpha_2} - E^{n+1}_{-\alpha_1-\alpha_2}) + (E^n_{\alpha_2+\alpha_3} - E^{n+1}_{-\alpha_2-\alpha_3}),\\
&\quad (E^n_{\alpha_1+\alpha_2+\alpha_3} - E^{n+1}_{-\alpha_1-\alpha_2-\alpha_3}) + (E^n_{\alpha_2} - E^{n+1}_{-\alpha_2})\};\\
\hat{\mathcal{H}}_3 \;&=\; \{(E^n_{\alpha_1} - E^n_{-\alpha_1}) + (E^n_{\alpha_3} - E^n_{-\alpha_3}),\ (E^n_{\alpha_1+\alpha_2} + E^{n+1}_{-\alpha_1-\alpha_2}) - (E^n_{\alpha_2+\alpha_3} + E^{n+1}_{-\alpha_2-\alpha_3}),\\
&\quad (E^n_{\alpha_1+\alpha_2+\alpha_3} - E^{n+1}_{-\alpha_1-\alpha_2-\alpha_3}) - (E^n_{\alpha_2} - E^{n+1}_{-\alpha_2})\};\\
\hat{\mathcal{H}}_4 \;&=\; \{(E^n_{\alpha_1} - E^n_{-\alpha_1}) - (E^n_{\alpha_3} - E^n_{-\alpha_3}),\ (E^n_{\alpha_1+\alpha_2} - E^{n+1}_{-\alpha_1-\alpha_2}) - (E^n_{\alpha_2+\alpha_3} - E^{n+1}_{-\alpha_2-\alpha_3}),\\
&\quad (E^n_{\alpha_1+\alpha_2+\alpha_3} + E^{n+1}_{-\alpha_1-\alpha_2-\alpha_3}) - (E^n_{\alpha_2} + E^{n+1}_{-\alpha_2})\}.
\end{aligned} \tag{44}$$

The corresponding grading operator is (see (3))

$$Q_{\mathbf{s}_{\rm p}} = \frac{2\lambda_2}{\alpha_2^2}\cdot H + 2d \tag{45}$$

and the generators are split into odd and even eigenvalues, i.e. introducing the subspaces

$$\begin{aligned}\hat{\mathcal{G}}_{2n} &\equiv \{E^n_{\alpha_1}, E^n_{-\alpha_1}; E^n_{\alpha_3}, E^n_{-\alpha_3}; H^n_a \ (a=1,2,3)\}, \\ \hat{\mathcal{G}}_{2n+1} &\equiv \{E^n_{\alpha_1+\alpha_2}, E^{n+1}_{-\alpha_1-\alpha_2}; E^n_{\alpha_2+\alpha_3}, E^{n+1}_{-\alpha_2-\alpha_3}, E^n_{\alpha_1+\alpha_2+\alpha_3}, E^{n+1}_{-\alpha_1-\alpha_2-\alpha_3}; E^n_{\alpha_2}, E^{n+1}_{-\alpha_2}\},\end{aligned} \tag{46}$$

one has

$$\begin{aligned}[Q_{\mathbf{s}_{\rm p}}\,,\,\hat{\mathcal{G}}_{2n}] &= 2n\,\hat{\mathcal{G}}_{2n} \\ [Q_{\mathbf{s}_{\rm p}}\,,\,\hat{\mathcal{G}}_{2n+1}] &= (2n+1)\,\hat{\mathcal{G}}_{2n+1}\end{aligned} \tag{47}$$

3.2 The case A_8

As usual we denote the simple roots of A_8 as α_a, $a=1,2,\ldots,8$, such that α_1 and α_8 correspond to the end points of the Dynkin diagram of A_8. Again we consider the gradation $\mathbf{s}_{\rm p}$ in (37), i.e.

$$\mathbf{s}_{\rm p} = (1,0,0,1,0,0,1,0,0) \tag{48}$$

The corresponding grading operator (3) is then

$$Q_{\mathbf{s}_{\rm p}} = \frac{2\lambda_3}{\alpha_3^2}\cdot H + \frac{2\lambda_6}{\alpha_6^2}\cdot H + 3d \tag{49}$$

The affine subspace $\mathcal{H}_0$ in (36) is again generated by H^n_a, $a=1,2,\ldots,8$, and the generators of the 9 affine subspaces $\mathcal{H}_{\mathbf{L}}$ are linear combinations of the following sets of step operators

$$\begin{aligned}\hat{\mathcal{G}}^{(1)}_{3n} &= \{E^n_{\alpha_1}, E^n_{\alpha_2}, E^n_{-\alpha_1-\alpha_2}; E^n_{\alpha_4}, E^n_{\alpha_5}, E^n_{-\alpha_4-\alpha_5}; E^n_{\alpha_7}, E^n_{\alpha_8}, E^n_{-\alpha_7-\alpha_8}\}, \\ \hat{\mathcal{G}}^{(2)}_{3n+1} &= \{E^n_{\alpha_2+\alpha_3}, E^n_{\alpha_4+\alpha_5+\alpha_6+\alpha_7+\alpha_8}, E^{n+1}_{-\alpha_2-\alpha_3-\alpha_4-\alpha_5-\alpha_6-\alpha_7-\alpha_8}; \\ &\quad E^n_{\alpha_3+\alpha_4}, E^n_{\alpha_5+\alpha_6}, E^{n+1}_{-\alpha_3-\alpha_4-\alpha_5-\alpha_6}; \\ &\quad E^n_{\alpha_6+\alpha_7}, E^n_{\alpha_1+\alpha_2+\alpha_3+\alpha_4+\alpha_5}, E^{n+1}_{-\alpha_1-\alpha_2-\alpha_3-\alpha_4-\alpha_5-\alpha_6-\alpha_7}\}, \\ \hat{\mathcal{G}}^{(3)}_{3n+1} &= \{E^n_{\alpha_1+\alpha_2+\alpha_3}, E^n_{\alpha_4+\alpha_5+\alpha_6}, E^{n+1}_{-\alpha_1-\alpha_2-\alpha_3-\alpha_4-\alpha_5-\alpha_6}; \\ &\quad E^n_{\alpha_2+\alpha_3+\alpha_4}, E^n_{\alpha_5+\alpha_6+\alpha_7}, E^{n+1}_{-\alpha_2-\alpha_3-\alpha_4-\alpha_5-\alpha_6-\alpha_7}; \\ &\quad E^n_{\alpha_3+\alpha_4+\alpha_5}, E^n_{\alpha_6+\alpha_7+\alpha_8}, E^{n+1}_{-\alpha_3-\alpha_4-\alpha_5-\alpha_6-\alpha_7-\alpha_8}\}, \\ \hat{\mathcal{G}}^{(4)}_{3n+1} &= \{E^n_{\alpha_1+\alpha_2+\alpha_3+\alpha_4}, E^n_{\alpha_5+\alpha_6+\alpha_7+\alpha_8}, E^{n+1}_{-\alpha_1-\alpha_2-\alpha_3-\alpha_4-\alpha_5-\alpha_6-\alpha_7-\alpha_8}; \\ &\quad E^n_{\alpha_2+\alpha_3+\alpha_4+\alpha_5}, E^n_{\alpha_6}, E^{n+1}_{-\alpha_2-\alpha_3-\alpha_4-\alpha_5-\alpha_6}; \\ &\quad E^n_{\alpha_3} + E^n_{\alpha_4+\alpha_5+\alpha_6+\alpha_7}, E^{n+1}_{-\alpha_3-\alpha_4-\alpha_5-\alpha_6-\alpha_7}\},\end{aligned} \tag{50}$$

and 4 other sets of elements which are conjugated to these by $E^n_{\pm\alpha} \to E^{-n}_{\mp\alpha}$. The lower indices on $\hat{\mathcal{G}}$'s are the eigenvalues w.r.t. the grading operator (49). Notice that each of the 8 subspaces $\hat{\mathcal{G}}$'s contains 9 elements. The generators of the 9 Heisenberg subalgebras $\mathcal{H}_{\mathbf{L}}$ will be linear combinations of the 9 elements of each one of those subspaces. And each one of the 8 elements of a given Heisenberg subalgebra comes from a different subspace $\hat{\mathcal{G}}^{(i)}$.

4 The case of G_2

The authors of ref. [3] considered also what they called *a root orthogonal decomposition* $\mathcal{ROD}$. In this case the algebra is decomposed as in (1) with the subspace $\mathcal{H}_0$ being the usual Cartan subalgebra and the remaining ones being generated by operators of the form $E_\alpha \pm E_{-\alpha}$, with α being a positive root of $\mathcal{G}$. Clearly any two roots, α and β appearing in a given subspace $\mathcal{H}_i$, must be such that $\pm\alpha \pm \beta$ are not roots.

The complex Lie algebra G_2 was shown, in ref. [3], to possess $\mathcal{ROD}$. Denote the roots of G_2 as α_1, α_2, $\alpha_1+\alpha_2$, $2\alpha_1+\alpha_2$, $3\alpha_1+\alpha_2$ and $3\alpha_1+2\alpha_2$, with α_1 and α_2 being the simple roots. The seven subspaces in the $\mathcal{ROD}$ are

$$
\begin{aligned}
\mathcal{H}_0 &= \{H_1\,,\, H_2\} \\
\mathcal{H}_1^{\pm} &= \{E_{\alpha_1} \pm E_{-\alpha_1}\,,\, E_{3\alpha_1+2\alpha_2} \pm E_{-3\alpha_1-2\alpha_2}\} \\
\mathcal{H}_2^{\pm} &= \{E_{\alpha_1+\alpha_2} \pm E_{-\alpha_1-\alpha_2}\,,\, E_{3\alpha_1+\alpha_2} \pm E_{-3\alpha_1-\alpha_2}\} \\
\mathcal{H}_3^{\pm} &= \{E_{\alpha_2} \pm E_{-\alpha_2}\,,\, E_{2\alpha_1+\alpha_2} \pm E_{-2\alpha_1-\alpha_2}\}
\end{aligned}
\tag{51}
$$

where H_a, $a=1,2$, and $E_{\pm\alpha}$ are the Cartan subalgebra generators and step operators respectivally in the Chevalley basis. Those subspaces are clearly abelian and orthogonal. In addition, that decomposition is also *multiplicative orthogonal* in the sense that for any i and j there exists a k such that $[\mathcal{H}_i\,,\,\mathcal{H}_j] \subset \mathcal{H}_k$.

In order to extend this decomposition to the affine Kac-Moody algebra $\hat{G}_2$ we have to find the gradings that respect the decompostion. Since the subspaces are made of linear combinations of positive and negative step operators associated to each root, the grading has to be such that for every positive root α there are integers $n_\pm$ such that $E_\alpha^{n_+}$ and $E_{-\alpha}^{n_-}$ have the same grades. From (3) and (4) we see that for a root $\alpha = m_1\alpha_1 + m_2\alpha_2$ we need $(n_- - n_+) = 2(m_1 s_1 + m_2 s_2)/N \equiv$ integer, where $N = s_0 + 3s_1 + 2s_2$ (since $\psi = 3\alpha_1 + 2\alpha_2$). For $\alpha \equiv \alpha_1$ we observe that this is impossible if $s_1 \neq 0$. Now taking $s_1 = 0$, we get for $\alpha \equiv m_1\alpha_1 + \alpha_2$ that $(n_- - n_+)$ is integer only if $s_0 = 0$ or then if $s_2 = 0$. For $\alpha \equiv 3\alpha_1 + 2\alpha_2$ these two possibilities are also allowed. So there are two gradations which respect the decomposition above, namely,

$$
\begin{aligned}
(s_0, s_1, s_2) &= (1,0,0) \qquad \text{homogeneous gradation} \\
(s_0, s_1, s_2) &= (0,0,1)
\end{aligned}
\tag{52}
$$

which are not equivalent, since there are no symmetries of the Dynkin diagram relating them. We shall refer to the second gradation above as Q_2 gradation.

We will consider here the gradation Q_2, since the same results can be easily translated to the homogeneous gradation. We define the generators

$$
\begin{aligned}
\hat{\mathcal{H}}_{0,1}^{(2n)} &\equiv (2H_1^n + 3H_2^n)/\sqrt{3} \\
\hat{\mathcal{H}}_{0,2}^{(2n)} &\equiv H_2^n \\
\hat{\mathcal{H}}_{1,1}^{\pm,(2n)} &\equiv (E_{\alpha_1}^n \pm E_{-\alpha_1}^n)/i^\sigma\sqrt{3} \\
\hat{\mathcal{H}}_{1,2}^{\pm,(2n)} &\equiv (E_{3\alpha_1+2\alpha_2}^{n-1} \pm E_{-3\alpha_1-2\alpha_2}^{n+1})/i^\sigma \\
\hat{\mathcal{H}}_{2,1}^{\pm,(2n+1)} &\equiv (E_{\alpha_1+\alpha_2}^n \pm E_{-\alpha_1-\alpha_2}^{n+1})/i^\sigma\sqrt{3} \\
\hat{\mathcal{H}}_{2,2}^{\pm,(2n+1)} &\equiv (E_{3\alpha_1+\alpha_2}^n \pm E_{-3\alpha_1-\alpha_2}^{n+1})/i^\sigma
\end{aligned}
$$

$$
\begin{aligned}
\hat{\mathcal{H}}_{3,1}^{\pm,(2n+1)} &\equiv (E_{\alpha_2}^n \pm E_{-\alpha_2}^{n+1})/i^\sigma \\
\hat{\mathcal{H}}_{3,2}^{\pm,(2n+1)} &\equiv (E_{2\alpha_1+\alpha_2}^n \pm E_{-2\alpha_1-\alpha_2}^{n+1})/i^\sigma\sqrt{3}
\end{aligned}
\tag{53}
$$

where $\sigma = 0$ for the $+$ sign and $\sigma = 1$ for the $-$ sign. These generators are all eigenvectors of the grading operator (see (3))

$$
Q_2 \equiv \lambda_2^v \cdot H + 2d \tag{54}
$$

and the eigenvalues are the numbers between parentheses in the upper indices in (53).

The affine Kac-Moody algebra $\hat{G}_2$ can then be decomposed into

$$
\hat{G}_2 = \hat{\mathcal{H}}_0 \oplus \hat{\mathcal{H}}_1^+ \oplus \hat{\mathcal{H}}_1^- \oplus \hat{\mathcal{H}}_2^+ \oplus \hat{\mathcal{H}}_2^- \oplus \hat{\mathcal{H}}_3^+ \oplus \hat{\mathcal{H}}_3^- \tag{55}
$$

with

$$
\begin{aligned}
\hat{\mathcal{H}}_0 &\equiv \{\hat{\mathcal{H}}_{0,a}^{(2n)} \, , \, a = 1, 2 \, , \, n \in \mathbb{Z}\} \\
\hat{\mathcal{H}}_1^\pm &\equiv \{\hat{\mathcal{H}}_{1,a}^{\pm,(2n)} \, , \, a = 1, 2 \, , \, n \in \mathbb{Z}\} \\
\hat{\mathcal{H}}_2^\pm &\equiv \{\hat{\mathcal{H}}_{2,a}^{\pm,(2n+1)} \, , \, a = 1, 2 \, , \, n \in \mathbb{Z}\} \\
\hat{\mathcal{H}}_3^\pm &\equiv \{\hat{\mathcal{H}}_{3,a}^{\pm,(2n+1)} \, , \, a = 1, 2 \, , \, n \in \mathbb{Z}\}
\end{aligned}
\tag{56}
$$

Using the bilinear form

$$
\mathrm{Tr}(H_a^m \, H_b^n) = \eta_{ab}\delta_{m+n,0} \qquad \mathrm{Tr}(E_\alpha^m \, E_\beta^n) = \frac{2}{\alpha^2}\delta_{\alpha+\beta,0}\delta_{m+n,0} \tag{57}
$$

where

$$
\eta_{ab} \equiv \frac{2}{\alpha_a^2} K_{ab} = \begin{pmatrix} 6 & -3 \\ -3 & 2 \end{pmatrix} \tag{58}
$$

and where we have normalized the roots as $\alpha_1^2 = 2/3$ and $\alpha_2^2 = 2$, one can check that the above subspaces are orthogonal.

Using (16) one can check that

$$
\begin{aligned}
[\hat{\mathcal{H}}_{0,a}^{(2m)} \, , \, \hat{\mathcal{H}}_{0,b}^{(2n)}] &= C\, 2m \, \delta_{a,b} \, \delta_{m+n,0} \\
[\hat{\mathcal{H}}_{1,a}^{\epsilon,(2m)} \, , \, \hat{\mathcal{H}}_{1,b}^{\epsilon,(2n)}] &= C\, 2m \, \delta_{a,b} \, \delta_{m+n,0} \\
[\hat{\mathcal{H}}_{2,a}^{\epsilon,(2m+1)} \, , \, \hat{\mathcal{H}}_{2,b}^{\epsilon,(2n+1)}] &= C\, (2m+1) \, \delta_{a,b} \, \delta_{m+n+1,0} \\
[\hat{\mathcal{H}}_{3,a}^{\epsilon,(2m+1)} \, , \, \hat{\mathcal{H}}_{3,b}^{\epsilon,(2n+1)}] &= C\, (2m+1) \, \delta_{a,b} \, \delta_{m+n+1,0}
\end{aligned}
\tag{59}
$$

where $\epsilon \equiv \pm$ and $a, b = 1, 2$. Therefore we have seven orthogonal Heisenberg subalgebras.

In addition, using the relations (16) with the fact that $\epsilon(\alpha \, , \, \beta) = -\epsilon(-\alpha \, , \, -\beta)$, one can check that

$$
\begin{aligned}
[\hat{\mathcal{H}}_i^+ \, , \, \hat{\mathcal{H}}_i^-] &\subset \hat{\mathcal{H}}_0 \\
[\hat{\mathcal{H}}_0 \, , \, \hat{\mathcal{H}}_i^\pm] &\subset \hat{\mathcal{H}}_i^\mp \\
[\hat{\mathcal{H}}_i^{\epsilon_1} \, , \, \hat{\mathcal{H}}_j^{\epsilon_2}] &\subset \hat{\mathcal{H}}_k^{-\epsilon_1\epsilon_2}
\end{aligned}
\tag{60}
$$

where $i, j, k = 1, 2, 3$ and in the last relation $i \neq j \neq k$; $\epsilon_1, \epsilon_2 = \pm$.

5 Conclusion

In the present paper we studied the decomposability problem of the affine Lie algebras into the algebraic sum of pairwise orthogonal Heisenberg subalgebras. In particular, the Kostrikin–Kostrikin–Ufnarovskii theorem about the orthogonal decomposition of some complex simple Lie algebras can be extended to the case of the corresponding affine Lie algebras endowed with the homogeneous gradation. However, the most interesting situation arises when other gradations also respect such a rigid decomposition, and we investigated this possibility in detail by the examples of the Lie algebras of type $A_r^{(1)}$ and $G_2^{(1)}$. Unfortunately, as in the finite–dimensional case, the consideration itself crucially depends on an explicit realization of the algebras in question, and is performed separately for various types of them. So, it is very interesting to find an uniform formulation for all algebras which admit the orthogonal (and multiplicative) decomposition.

An interesting issue stimulated by the existence of such decompositions concerns the representation theory of these special affine Kac-Moody algebras. As is well known, the Heisenberg subalgebra is the algebra of oscillators in quantum field theory. So, one could conjecture that representations for these algebras could be constructed introducing a set of oscillators for each Heisenberg subalgebra in the decomposition. These oscillators however could not be independent since the Heisenberg subalgebras do not commute among themselves. Their commutation relations, like the ones we calculated in the case of A_{p-1} and G_2, could give hints on how to relate those oscillators. These considerations could be related to previous attempts to use division algebras, Jordan algebras, dependent fermions etc, in the construction of representations of Kac-Moody algebras, see e.g. [9].

We suspect that the orthogonal decomposition presented in section 2 is relevant for studying the completeness problem of the set of the soliton solutions to the affine Toda theories obtained in the framework of the technique discovered in [10], especially for their nonabelian versions discussed in [11]. Moreover, the orthogonal decomposition being closely related to the finest gradation of the algebras, could be relevant for a construction of some multidimensional integrable nonlinear systems in the framework of the Lie group algebraic approach [12]. It would be also interesting to study the analogue of this decomposition for the algebras of diffeomorphisms groups, as e.g. the continuous limit of the A_r algebra in the cyclotomic basis to that of the area preserving diffeomorphisms on a two dimensional manifold. That could be relevant for some problems related to an affinization of the membrane manifolds in the context of continuous limits of the gauge theories, 3-d gravity, etc.

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